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Local fractional derivatives and fractal functions of several variables

Kiran M. Kolwankar* and Anil D. Gangal†

Department of Physics, University of Pune, Pune 411 007, India.

The notion of a local fractional derivative (LFD) was introduced recently for functions of a single variable. LFD was shown to be useful in studying fractional differentiability properties of fractal and multifractal functions. It was demonstrated that the local Hölder exponent/ dimension was directly related to the maximum order for which LFD existed. We have extended this definition to directional-LFD for functions of many variables and demonstrated its utility with the help of simple examples.

I. INTRODUCTION

Fractal and multifractal functions and the corresponding curves or surfaces are found in numerous places in nonlinear and nonequillibrium phenomenon. For example, isoscalar surfaces for advected scalars in certain turbulence problems [1,2], typical Feynman [3,4] and Brownian paths [5,6], attractors of some dynamical systems [7] are, among many others, examples of occurence of continuous but highly irregular (nondifferentiable) curves and surfaces. Velocity field of a turbulent fluid [8] at low viscosity is a well-known example of a multifractal function. Ordinary calculus is inadequate to characterize and handle such curves and surfaces. Some recent papers [9–12] indicate a connection between fractional calculus [13–15] and fractal structure [5,6] or fractal processes [16–18]. However the precise nature of the connection between the dimension of the graph of a fractal curve and fractional differentiability properties was recognized only recently. A new notion of *local fractional derivative* (LFD) was introduced [19,20] to study fractional differentiability properties of irregular functions. An intresting observation of this work was that the local Hölder exponent/ dimension was related to the maximum order for which LFD existed. In this paper we briefly review the concept of LFD as applied to a function of one variable and generalize it for functions of several variables.

An irregular function of one variable is best characterized locally by a *Hölder exponent*. We will use the following general definition of the Hölder exponent which has been used by various authors [21,22] recently. The exponent $h(y)$ of a function f at y is given by h such that there exists a polynomial $P_n(x)$ of order n, where n is the largest integer smaller than h, and

$$|f(x) - P_n(x-y)| = O(|x-y|^h), \tag{1}$$

for x in the neighbourhood of y. This definition serves to classify the behavior of the function at y.

Fractional calculus [13–15] is a study which deals with generalization of differentiation and integration to fractional orders. There are number of ways (not neccesarily equivalent) of defining fractional derivatives and integrations. We use the Riemann-Liouville definition. We begin by recalling the Riemann-Liouville definition of fractional integral of a real function, which is given by [13,15]

$$\frac{d^q f(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{f(y)}{(x-y)^{q+1}} dy \quad \text{for} \quad q < 0, \tag{2}$$

where the lower limit a is some real number and of the fractional derivative

$$\frac{d^q f(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_a^x \frac{f(y)}{(x-y)^{q-n+1}} dy \tag{3}$$

for $n-1 < q < n$. Fractional derivative of a simple function $f(x) = x^p \quad p > -1$ is given by [13,15]

$$\frac{d^q x^p}{dx^q} = \frac{\Gamma(p+1)}{\Gamma(p-q+1)} x^{p-q} \quad \text{for} \quad p > -1. \tag{4}$$

Further the fractional derivative has the interesting property (see ref [13]), viz,

$$\frac{d^q f(\beta x)}{dx^q} = \beta^q \frac{d^q f(\beta x)}{d(\beta x)^q} \tag{5}$$

which makes it suitable for the study of scaling. Note the nonlocal character of the fractional integral and derivative in the equations (2) and (3) respectively. Also it is clear from equation (4) that unlike in the case of integer derivatives the fractional derivative of a constant is not zero in general. These two features make the extraction of scaling information somewhat difficult. The problems were overcome by the introduction of LFD in [19]. In the following section we briefly review the notion of LFD for the real valued functions of real variable. In the section III we generalize this definition to real valued functions of many variables and demonstrate it with the help of some simple examples.

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*email: kirkol@physics.unipune.ernet.in
†email: adg@physics.unipune.ernet.in

II. LOCAL FRACTIONAL DIFFERENTIABILITY

Unfortunately, as noted in section I, fractional derivatives are not local in nature. On the other hand it is desirable and occasionally crucial to have local character in wide range of applications ranging from the structure of differentiable manifolds to various physical models. Secondly the fractional derivative of a constant is not zero, consequently the magnitude of the fractional derivative changes with the addition of a constant. The appropriate new notion of fractional differentiability must bypass the hindrance due to these two properties. These difficulties were remedied by introducing the notion LFD in [19] as follows:

Definition 1 *If, for a function $f : [0,1] \rightarrow I\!R$, the limit*

$$I\!D^q f(y) = \lim_{x\rightarrow y} \frac{d^q(f(x)-f(y))}{d(x-y)^q} \tag{6}$$

exists and is finite, then we say that the local fractional derivative (LFD) of order q $(0<q<1)$, at $x=y$, exists.

In the above definition the lower limit y is treated as a constant. The subtraction of $f(y)$ corrects for the fact that the fractional derivative of a constant is not zero. Whereas the limit as $x \rightarrow y$ is taken to remove the nonlocal content. Advantage of defining local fractional derivative in this manner lies in its local nature and hence allowing the study of pointwise behaviour of functions. This will be seen more clearly after the development of Taylor series below.

Definition 2 *We define critical order α, at y, as*

$$\alpha(y) = Sup\{q \mid \text{all LFDs of order less than } q \text{ exist at } y\}.$$

These definitions were subsequently generalized [20] for $q>1$ as follows.

Definition 3 *If, for a function $f : [0,1] \rightarrow I\!R$, the limit*

$$I\!D^q f(y) = \lim_{x\rightarrow y} \frac{d^q(f(x)-\sum_{n=0}^{N}\frac{f^{(n)}(y)}{\Gamma(n+1)}(x-y)^n)}{[d(x-y)]^q} \tag{7}$$

exists and is finite, where N is the largest integer for which N^{th} derivative of $f(x)$ at y exists and is finite, then we say that the local fractional derivative (LFD) of order q $(N<q\leq N+1)$, at $x=y$, exists.

We consider this as the generalization of the local derivative for order greater than one. Note that when q is a positive integer ordinary derivatives are recovered. The definition of the critical order remains the same since, for $q<1$, (6) and (7) agree. This definition extends the applicability of LFD to C^1-functions which are still irregular due to the nonexistence of some higher order derivative (i.e. belong to class C^γ, $\gamma>1$). For example the critical order of $f(x)=a+bx+c|x|^\gamma$, $\gamma>1$, at origin, according to definitions 2 and 3 is γ. It is clear that the critical order of the C^∞ function is ∞.

In [19] it was shown that the Weierstrass nowhere differentiable function, given by

$$W_\lambda(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} sin\lambda^k t, \tag{8}$$

where $\lambda>1$, $1<s<2$ and t real, is continuously locally fractional differentiable for orders below $2-s$ and not for orders between $2-s$ and one. This implies that the critical order of this function is $2-s$ at all points. Interesting consequence of this result is the relation between box dimension s [6] of the graph of the function and the critical order. This result has for the first time given a direct relation between the differentiability property and the dimension. In fact this observation was consolidated into a general result showing equivalence between critical order and the local Hölder exponent of any continuous function. The LFD was further shown to be useful in the study of pointwise behaviour of multifractal functions and for unmasking the singularities masked by stronger singularities.

Whenever it is required to distinguish between limits from right and left sides we can write the definition for LFD in the following form.

$$I\!D^q_{\pm} f(y) = \lim_{x\rightarrow y^{\pm}} \frac{d^q \widetilde{F}_N(x,y)}{[d\pm(x-y)]^q} \tag{9}$$

The importance of the notion of LFD lies in the fact that it appears naturally in the fractional Taylor's series with a remainder term for a real function f, given by,

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(y)}{\Gamma(n+1)}\Delta^n + \frac{I\!D^q_+ f(y)}{\Gamma(q+1)}\Delta^q + R_q(y,\Delta) \tag{10}$$

where $x-y=\Delta>0$ and $R_q(y,\Delta)$ is a remainder given by

$$R_q(y,\Delta) = \frac{1}{\Gamma(q+1)}\int_0^\Delta \frac{dF(y,t;q,N)}{dt}(\Delta-t)^q dt \tag{11}$$

and

$$F(y,\Delta;q,N) = \frac{d^q(f(x)-\sum_{n=0}^{N}\frac{f^{(n)}(y)}{\Gamma(n+1)}\Delta^n)}{[d\Delta]^q} \tag{12}$$

We note that the local fractional derivative as defined above (not just fractional derivative), along with the first N derivatives, provides an approximation of $f(x)$ in the vicinity of y. We further remark that the terms on the RHS of eqn(10) are nontrivial and finite only in the case when q equals α, the critical order. Osler [23] constructed a fractional Taylor series using usual (Riemann-Liouville) fractional derivatives which was applicable only to analytic function. Further Osler's formulation involves terms

with negative orders also and hence is not suitable for approximating schemes. When $\Delta < 0$, a similar expansion can be written for $I\!\!D^q_- f(y)$ by replacing Δ by $-\Delta$.

When $0 < q < 1$ we get as a special case

$$f(x) = f(y) + \frac{I\!\!D^q f(y)}{\Gamma(q+1)}(x-y)^q + Remainder \qquad (13)$$

provided the RHS exists. If we set q equal to one in equation (13) one gets the equation of the tangent. All the curves passing through a point y and having same the tangent, form an equivalence class (which is modelled by a linear behavior). Analogously all the functions (curves) with the same critical order α and the same $I\!\!D^\alpha$ will form an equivalence class modeled by the power law x^α. This is how one may generalize the geometric interpretation of derivatives in terms of tangents. This observation is useful in the approximation of an irregular function by a piecewise smooth (scaling) function. One may recognize the introduction of such equivalence classes as a starting point for fractional differential geometry.

III. GENERALIZATION TO HIGHER DIMENSIONAL FUNCTIONS

The definition of the Local fractional derivative can be generalized for higher dimensional functions in the following manner.

Consider a function $f : I\!\!R^n \to I\!\!R$. We define

$$\Phi(\mathbf{y}, t) = f(\mathbf{y} + \mathbf{v}t) - f(\mathbf{y}), \quad \mathbf{v} \in I\!\!R^n, \quad t \in I\!\!R. \qquad (14)$$

Then the directional-LFD of f at $\mathbf{y}$ of order q, $0 < q < 1$, in the direction $\mathbf{v}$ is given (provided it exists) by

$$I\!\!D^q_{\mathbf{v}} f(\mathbf{y}) = \frac{d^q \Phi(\mathbf{y}, t)}{dt^q}\Big|_{t=0} \qquad (15)$$

where the RHS involves the usual fractional derivative of equation (3). The directional-LFDs along the unit vector $\mathbf{e}_i$ will be called i^{th} partial-LFD.

Now let us consider two examples.

Example 1: Let

$$W_\lambda(\mathbf{x}) = W_\lambda(x, y) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} sin\lambda^k(x+y), \qquad (16)$$

with $\lambda > 1$ and $1 < s < 2$. Let $\mathbf{v} = (v_x, v_y)$ be a unit 2-vector. Then

$$\begin{aligned} W_\lambda(\mathbf{x} + \mathbf{vt}) &= W_\lambda(x + v_x t, y + v_y t) \\ &= \sum_{k=1}^{\infty} \lambda^{(s-2)k} sin\lambda^k(x + v_x t + y + v_y t). \end{aligned}$$

$$\begin{aligned} \Phi(\mathbf{x}, t) &= W_\lambda(\mathbf{x} + \mathbf{vt}) - W_\lambda(\mathbf{x}) \\ &= \sum_{k=1}^{\infty} \lambda^{(s-2)k} [sin\lambda^k(x + v_x t + y + v_y t) \\ &\qquad - sin\lambda^k(x+y)] \end{aligned}$$

If we choose $y = 0$, i.e., we examine fractional differentiability at a point on x axis and if we choose $v_y = 0$, i.e., we are looking at LFD in the direction of increasing x then we get

$$\Phi(\mathbf{x}, t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} [sin\lambda^k(x + v_x t) - sin\lambda^k(x)]$$

which is known [19] to have critical order $2 - s$.

If we keep $y = 0$ but keep v_x and v_y non-zero, i.e., if we examine fractional differentiability at a point on x-axis but in any direction then we get

$$\Phi(\mathbf{x}, t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} [sin\lambda^k(x + v_x t + v_y t) - sin\lambda^k(x)]$$

This again has critical order $2-s$, except when $v_x = -v_y$ (in which case it is ∞ since the function Φ identically vanishes).

In fact it can be shown that the given function has $2-s$ as critical order at any point and in any direction except the direction with $v_x = -v_y$.

Example 2: Another example we consider is

$$W_\lambda(\mathbf{x}) = W_\lambda(x, y) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} sin\lambda^k(xy), \qquad (17)$$

where $\lambda > 1$ and $1 < s < 2$.

$$\begin{aligned} W_\lambda(\mathbf{x} + \mathbf{vt}) &= W_\lambda(x + v_x t, y + v_y t) \\ &= \sum_{k=1}^{\infty} \lambda^{(s-2)k} sin(\lambda^k(x + v_x t)(y + v_y t)). \end{aligned}$$

$$\begin{aligned} \Phi(\mathbf{x}, t) &= W_\lambda(\mathbf{x} + \mathbf{vt}) - W_\lambda(\mathbf{x}) \\ &= \sum_{k=1}^{\infty} \lambda^{(s-2)k} [sin(\lambda^k(xy + yv_x t + xv_y t + v_x v_y t^2)) \\ &\qquad - sin(\lambda^k(xy))] \end{aligned}$$

Now if we choose $y = 0$ and $v_y = 0$ then the critical order is ∞.

If we choose $y = 0$ but v_x and v_y non-zero then we get

$$\Phi(\mathbf{x}, t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} [sin(\lambda^k(xv_y t + v_x v_y t^2))]$$

Therefore using results of [19] the critical order along any other direction $\mathbf{v}$, at a point on x-axis is seen to be $2 - s$.

IV. CONCLUSIONS

The usefulness of the notion of LFD was pointed out in [19,20] where the considerations were restricted to functions of one variable only. It allows us to quantify the loss of differentiability of fractal and multifractal functions. The larger the irregularity of the functions the smaller is the extent of differentiability and smaller is the value of the Hölder exponent. Local Taylor series expansions provide a way to approximate irregular functions by a piecewise scaling functions. In the present paper we have demonstrated that it is possible to carry the same theme even in the multidimensional case. In particular, the Hölder exponents in any direction are related to the critical order of the corresponding directional-LFD. We note that, whereas a one dimensional result is useful in studying irregular signals, the results here may have utility in image processing where one can characterize and classify singularities in the image data. We note that it is possible to write a multivariable fractional Taylor series which can be used for approximations and modelling of multivariable multiscaling functions. This development will be reported elsewhere. Further these considerations provide a way for formulating fractional differential geometry for fractal surfaces.

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